UDC 539.3

ACTION OF AN ELLIPTIC STAMP MOVING AT A CONSTANT SPEED ON AN ELASTIC HALF-SPACE

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The action of a rigid stamp moving at a constant speed, on the boundary of an elastic half-space, is investigated. It is assumed that the frictional forces between the stamp and the surface of the half-space are absent. The integral equation obtained in [1] yields formulas for the pressure, for the case when the area of contact between the stamp and the half-space has an elliptic form.

1. Expansion of the kernel of integral equation into a power series. The author obtained in [1] an integral equation for determining the pressure under a rigid stamp moving at a constant speed along the x-axis. The equation written in the x, y, z-coordinate system attached to the moving stamp has the form

$$w(x, y) = \iint_{\Omega} K(x-\xi, y-\eta)\, q(\xi, \eta)\, d\xi\, d\eta \tag{1.1}$$

$$K(x-\xi, y-\eta) = \frac{k_2 (1 + r_1 r_2^{-1} \sqrt{S_3})\, r_1 r_2^{-1} \sqrt{S_3}}{2\pi\mu\, [4\gamma k_2^2 r^2 r_2^{-2} + (1 - k_2^2 r^2 r_2^{-2})(1 + r_1 r_2^{-1}\sqrt{S_3})]\, r_2}$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2, \quad r_1 = \sqrt{k_1^2 (x-\xi)^2 + (y-\eta)^2}$$

$$r_2 = \sqrt{k_2^2 (x-\xi)^2 + (y-\eta)^2}, \quad S_3 = k_2^2 k_1^{-2}$$

$$k_1^{-2} = 1 - c^2 c_1^{-2}, \quad k_2^{-2} = 1 - c^2 c_2^{-2}, \quad \gamma = (\lambda + \mu)/(\lambda + 2\mu)$$

where c_1 and c_2 are the longitudinal and transverse velocities of the waves generated by the stamp moving at the speed c.

To solve the equation we expand the kernel into a power series in

$$\varkappa = (y-\eta)^2 r^{-2} c^2 c_2^{-2} \tag{1.2}$$

Using the last four relations of (1.1), we obtain the following expression for the kernel of (1.1):

$$K(x-\xi, y-\eta) = \frac{1}{2\pi\mu\, r} K^*(\varkappa) \tag{1.3}$$

$$K(\varkappa) = \frac{[1 - \gamma + \gamma(1-\varkappa)^{-1} + B]\,(1-\varkappa)^{-1/2}}{[1 + (1-\varkappa)^{-1}(4\gamma - 1 - B\varkappa)]}$$

$$B = [1 - (1-\gamma)\varkappa]^{1/2} (1-\varkappa)^{-1/2}$$

Expanding the expressions of the form $(1 - \varkappa)^m$ in (1.3) into binomial series, we obtain the following expression for the kernel of (1.1):

$$K(x-\xi, y-\eta) = \sum_{\alpha=0}^{\infty} A_\alpha \frac{(y-\eta)^{2\alpha}}{r^{2\alpha+1}} \qquad (1.4)$$

$$A_0 = \frac{1}{4\pi\mu\gamma}, \quad A_1 = \frac{1}{8\pi\mu}\left(\frac{1}{\gamma^2} - \frac{1}{\gamma} + \frac{3}{2}\right)\frac{c^2}{c_2^2}$$

$$\alpha \geqslant 2, \quad A_\alpha = \frac{1}{8\pi\mu\gamma}\left\{[2\gamma\alpha - (1-\gamma)^\alpha - 1]\frac{(2\alpha-3)!!}{(2\alpha)!!} + \right.$$

$$\frac{4\pi\mu}{(c^2c_2^{-2})^{\alpha-1}} A_{\alpha-1} - 2\pi\mu \sum_{i=0}^{\alpha-2}\sum_{\beta=0}^{i+1} \frac{A_{\alpha-i-2}}{(c^2c_2^{-2})^{\alpha-i-2}} (1-\gamma)^\beta \times$$

$$\left.\frac{(2\beta-3)!!}{(2\beta)!!}\,\frac{(2i-2\beta+1)!!}{(2i-2\beta+2)!!}\right\}\left(\frac{c^2}{c_2^2}\right)^\alpha$$

$$(\alpha = 0,\ (-3)!! = (-1)!! = 0!! = 1)$$

The series (1.4) converges uniformly to $K(x-\xi, y-\eta)$ for $c < c_3$ where c_3 is the velocity of the Rayleigh surface wave. The latter velocity is given by the equation

$$(2 - c^2c_2^{-2})^4 = 16\,(1 - c^2c_2^{-2})\,(1 - c^2c_1^{-2}) \qquad (1.5)$$

which is obtained by equating the denominator in (1.3) to zero under the condition that $(y-\eta)^2r^{-2} = 1$. The relation (1.5) is also known in seismology (see e.g. [2, 3]).

Using the method of induction, we can obtain the following estimate for the coefficients A_α of (1.4):

$$A_\alpha < N\left(\frac{B}{\gamma}\right)^\alpha\left(\frac{c^2}{c_2^2}\right)^\alpha, \quad N = \text{const}, \quad \frac{1}{2(1-\gamma)} < B < \infty$$

where it should be remembered that $^1/_2 < \gamma < 1$.

2. Impressing a stamp of elliptic cross section into an elastic half-space. If

$$w(x, y) = \sum_{i=0}^{k}\sum_{j=0}^{l} b_{ij}x^iy^j \qquad (2.1)$$

$$(k + l = n,\ b_{ij} = \text{const})$$

then the solution of the integral equation

$$w(x, y) = \iint_\Omega q(\xi, \eta) \sum_{\alpha=0}^{\infty} A_\alpha \frac{(y-\eta)^{2\alpha}}{R^{2\alpha+1}}\, d\xi\, d\eta \qquad (2.2)$$

for an elliptic region of contact Ω (with the semiaxes a and b) has the form

$$q(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l} a_{ij} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} \qquad (2.3)$$

$$(k + l = n,\ a_{ij} = \mathrm{const},\ R = \sqrt{(x-\xi)^2 + (y-\eta)^2})$$

Assuming that the solution of (2.2) is unique, we shall prove the validity of (2.3), arriving at the same time at the method of obtaining a relation connecting the coefficients a_{ij} with b_{ij}.

We shall show that the integral $J(x, y)$ equal to the right hand side of (2.2) is an n-th degree polynomial, if $q(\xi, \eta)$ has the form (2.3), and we shall use the method given in [4, 5] to prove this.

Passing to the polar coordinates (see Fig. 1) $\xi = x + R\cos\varphi,\ \eta = y + R\sin\varphi$, we obtain , from (2.2) and (2.3),

$$J(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l} a_{ij} \sum_{\alpha=0}^{\infty} A_\alpha \int_0^{2\pi} d\varphi \int_0^{R_0(\varphi)} (x + R\cos\varphi)^i \times \qquad (2.4)$$

$$(y + R\sin\varphi)^j \sin^{2\alpha}\varphi \left[N + \frac{M^2}{L}\right]^{-1/2} \left[1 - \left(\frac{M + RL}{\sqrt{NL + M^2}}\right)^2\right]^{-1/2} dR$$

$$L(\varphi) = a^{-2}\cos^2\varphi + b^{-2}\sin^2\varphi, \quad M(\varphi) = a^{-2}x\cos\varphi + b^{-2} \times$$
$$y\sin\varphi$$
$$N = 1 - x^2/a^2 - y^2/b^2$$

Let us consider a point A within the ellipse, and a point A'' at its boundary. Let $R_0(\varphi)$ be the distance between the points A and A'' . Since the coordinates of A'' satisfy the relations

$$x_0^2/a^2 + y_0^2/b^2 = 1 \qquad (2.5)$$
$$x_0 = x + R_0\cos\varphi$$
$$y_0 = y + R_0\sin\varphi$$

the distance R_0 is given by the expression

$$R_0(\varphi) = (-M(\varphi) + \sqrt{M^2(\varphi) - NL(\varphi)})/L(\varphi) \qquad (2.6)$$

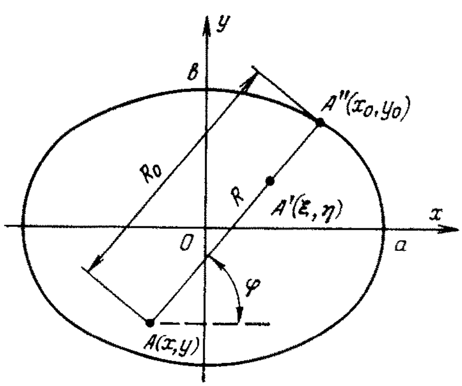

Fig. 1

Here $N > 0$ since $A(x, y)$ lies within the ellipse.

Since $0 \leqslant R \leqslant R_0(\varphi)$, $L(\varphi) > 0$, we have

$$-1 < \frac{M}{K} \leqslant \frac{M + RL}{K} \leqslant \frac{M + R_0 L}{K} = 1 \qquad (K = \sqrt{M^2 + NL})$$

Performing now the change of variable

$$\cos\theta = (M + RL) / K \qquad (0 \leqslant \theta \leqslant \pi)$$

we can rewrite (2.4) in the form

$$J(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l} a_{ij} \sum_{\alpha=0}^{\infty} A_\alpha \int_0^{2\pi} d\varphi \int_0^{\theta(\varphi)} J_{ij}\, d\theta \tag{2.7}$$

$$J_{ij}(\varphi, \theta) = (x + K\cos\varphi\cos\theta - L^{-1} M \cos\varphi)^i \times$$
$$(y + K\sin\varphi\cos\theta - L^{-1} M \sin\varphi)^j L^{-1/2} \sin^{2\alpha}\varphi$$

where $\theta(\varphi)$ denotes the value of θ at $R = 0$. Taking into account the following relation obtained from (2.4) and (2.7):

$$\int_\pi^{2\pi} d\varphi \int_0^{\theta(\varphi)} J_{ij}(\varphi, \theta(\varphi))\, d\theta = \int_0^{\pi} d\psi \int_0^{\pi-\theta(\psi)} J_{ij}(\psi, \theta(\psi))\, d\psi$$

we write the expression (2.7) in the form

$$J(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l} a_{ij} \sum_{\alpha=0}^{\infty} A_\alpha \int_0^{\pi} d\varphi \int_0^{\theta(\varphi)} J_{ij}(\varphi, \theta(\varphi))\, d\theta = \tag{2.8}$$
$$\sum_{i=0}^{k} \sum_{j=0}^{l} a_{ij} \sum_{\alpha=0}^{\infty} A_\alpha \int_0^{\pi} d\varphi \int_0^{\pi} J_{ij}(\varphi, \theta)\, d\theta$$

Integration with respect to θ reduces (2.8) to

$$J(x, y) = 2 \sum_{i=0}^{k} \sum_{j=0}^{l} a_{ij} \sum_{r=0}^{i} \sum_{s=0}^{j} C_i^r C_j^s \times \tag{2.9}$$
$$B\left(\frac{r+s+1}{2}, \frac{1}{2}\right) (1 - e^2)^{(r-s+2j+1)/2} \times$$
$$\sum_{q=0}^{(r+s)/2} C_{(r+s)/2}^q (-1)^{(r-s)/2-q} a^{2q+1} \sum_{p=0}^{i+j-2q} C_{i+j-2}^p \times$$
$$x^{i+j-2q-p} y^p \sum_{\alpha=0}^{\infty} A_\alpha S_{2\alpha,\, i-q+(j-p+s-r)/2,\, (j+p-s+r)/2}$$
$$S_{2\alpha, m, n} = \int_0^{\pi/2} \frac{\cos^{2m+2\alpha}\varphi \sin^{2n}\varphi\, d\varphi}{(1 - e^2\sin^2\varphi)^{m+n+1/2}}$$

where $r + s$ and $j + p$ are even numbers and e is the eccentricity of the elliptical region Ω. The quantities $S_{2\alpha, m, n}$ are given in terms of the complete elliptic integrals, and the expression (2.9) shows that $J(x, y)$ is indeed an n-th degree polynomial.

Substituting (2.9) into the right hand side of (2.2), we obtain a relation containing in its left and right hand sides n-th degree polynomials dependent of x and y. Equating the coefficients of like powers in x and y appearing in both sides of this relation, we obtain a system of $(n+1)(n+2)/2$ linear algebraic equations for determining the coefficients a_{ij} in terms of b_{ij}. Solving this system we obtain, in accordance with (2.3), a solution of the integral equation (2.2) for the case (2.1), for the elliptical region Ω.

The force P and the moments M_x, M_y are determined by formulas:

$$P = \iint_{\Omega} q(x, y)\, dx\, dy \tag{2.10}$$

$$M_x = \iint_{\Omega} y q(x, y)\, dx\, dy, \quad M_y = \iint_{\Omega} x q(x, y)\, dx\, dy$$

The solution (2.3) becomes infinite at the boundary of the elliptical region of contact Ω. We can, however, obtain a solution of (2.2) which will vanish at the boundary of the region of contact Ω. This solution has the form

$$q(x, y) = \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} a_{ij} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} \tag{2.11}$$

The proof of the validity of (2.11) is similar to that given above. In this case the relations connecting the coefficients a_{ij} and b_{ij} are obtained from

$$\sum_{i=0}^{k-1} \sum_{j=0}^{l-1} b_{ij} x^i y^j = 2 \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} a_{ij} \times \tag{2.12}$$

$$\sum_{r=0}^{i} \sum_{s=0}^{j} C_i^r C_j^s B\left(\frac{r+s+1}{2}, \frac{3}{2}\right) (1 - e^2)^{(r-s+2j+1)/2} \times$$

$$\sum_{q=0}^{(r+s+2)/2} C_{(r+s+2)/2}^{q} (-1)^{(r-s)/2-q+1} a^{2q-1} \sum_{p=0}^{i+j-2q+2} C_{i+j-2q+2}^{p} x^{i+j-2q-p+2} y^{p} \times$$

$$\sum_{\alpha=0}^{\infty} A_\alpha S_{2\alpha,\, i-q+1+(j-p+s-r)/2,\, (j+p-s+r)/2}$$

where $k + l = n$, $r + s$ and $j + p$ are even numbers.

Let us consider the case in which the pressure at the boundary of the region of contact is without limit. Let

$$w(x, y) = b_{00} + b_{10} x + b_{01} y \tag{2.13}$$

Then the polynomial (2.9) assumes the following form:

$$Z(x, y) = 2\pi b \sum_{\alpha=0}^{\infty} A_\alpha \left(S_{2\alpha, 0, 0} a_{00} + S_{2\alpha, 1, 0} a_{10} x + b^2 a^{-2} S_{2\alpha, 0, 1} a_{01} y\right) \tag{2.14}$$

Equating the free term and the coefficients accompanying the unknowns x and y in (2.13) with the corresponding quantities appearing in (2.14), we obtain the system

$$b_{i0} = 2\pi b \sum_{\alpha=0}^{\infty} A_\alpha S_{2\alpha,\, i,\, 0} a_{i0}, \quad i = 0, 1 \tag{2.15}$$

$$b_{01} = 2\pi b^3 a^{-2} \sum_{\alpha=0}^{\infty} A_\alpha S_{2\alpha,\, 0,\, 1} a_{01}$$

$$S_{2\alpha,\, i,\, j} = \int_0^{\pi/2} \frac{\cos^{2(\alpha+i)} \varphi \sin^{2j} \varphi \, d\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}$$

Solving (2.15) for a_{ij}, we obtain from (2.3) the expression for the pressure under the stamp in the form

$$q(x, y) = (2\pi b)^{-1} (b_{00} B_{00} + b_{10} B_{10} x + a^2 b^{-2} b_{01} B_{01} y)(1 - x^2/a^2 - y^2/b^2)^{-1/2} \tag{2.16}$$

$$B_{ij} = \left(\sum_{\alpha=0}^{\infty} A_\alpha S_{2\alpha,\, i,\, j} \right)^{-1}$$

The force and the moments are obtained from (2.10) and (2.16) in the form

$$P = a b_{00} B_{00}, \quad M_x = {}^1/_3 a^3 b_{01} B_{01}, \quad M_y = {}^1/_3 a^3 b_{10} B_{10}$$

Let us now consider the case in which the pressure at the boundary of the region of contact has a limit. Let

$$w(x, y) = b_{00} + b_{20} x^2 + b_{02} y^2 \tag{2.17}$$

In this case, using (2.12) we obtain the following system of equtions for determining the coefficients a_{ij} for the given semiaxes a and b of the elliptical region of contact:

$$b_{00} = \pi b \left[\sum_{\alpha=0}^{\infty} A_\alpha \left(S_{2\alpha,\, 0,\, 0} a_{00} + \frac{1}{4} b^2 S_{2\alpha,\, 0,\, 1} a_{20} + \frac{1}{4} b^2 S_{2\alpha,\, 1,\, 0} a_{02} \right) \right] \tag{2.18}$$

$$b_{20} = \pi b \left\{ \sum_{\alpha=0}^{\infty} A_\alpha \left[- a^{-2} S_{2\alpha,\, 1,\, 0} a_{00} + \left(S_{2\alpha,\, 2,\, 0} - \frac{1}{4} b^2 a^{-2} S_{2\alpha,\, 1,\, 1} \right) a_{20} + \right.\right.$$
$$\left.\left. \left(b^4 a^{-4} S_{2\alpha,\, 1,\, 1} + \frac{1}{4} b^2 a^{-2} S_{2\alpha,\, 2,\, 0} \right) a_{02} \right] \right\}$$

$$b_{02} = \pi b \left\{ \sum_{\alpha=0}^{\infty} A_\alpha \left[- a^{-2} S_{2\alpha,\, 0,\, 1} a_{00} + \left(S_{2\alpha,\, 1,\, 1} - \frac{1}{4} b^2 a^{-2} S_{2\alpha,\, 0,\, 2} \right) a_{20} + \right.\right.$$
$$\left.\left. \left(b^4 a^{-4} S_{2\alpha,\, 0,\, 2} - \frac{1}{4} b^2 a^{-2} S_{2\alpha,\, 1,\, 1} \right) a_{02} \right] \right\}$$

$$S_{2\alpha,\, i,\, j} = \int_0^{\pi/2} \frac{\cos^{2(\alpha+i)} \varphi \sin^{2j} \varphi \, d\varphi}{(1 - e^2 \sin^2 \varphi)^\beta}$$

$$j = 0, 2, \quad \beta = {}^3/_2; \quad j = 1, \quad \beta = {}^5/_2$$

Having determined a_{ij} we obtain from (2.11) the following expression for the pressure under the stamp:

$$q(x, y) = (a_{00} + a_{20}x^2 + a_{02}y^2)(1 - x^2 / a^2 - y^2 / b^2)^{-1/2} \tag{2.19}$$

The load acting on the stamp is obtained, using the integral (2.10), in the form

$$P = {}^2/_3 \pi ab\, [a_{00} + (a_{20}a^2 + a_{02}b^2) / 5] \tag{2.20}$$

Setting a_{20} and a_{02} in (2.18) — (2.20) equal to zero, we obtain a solution of the problem of impressing, into an elastic half-space, a moving parabolic stamp in which the area of contact varies with the force P applied.
The first equation of (2.18) yields the coefficient a_{00}, while the second the third equation of (2.18) are used to determine the semiaxes a and b of the elliptical region of contact, which are not known in this case.

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